Quantum Field Theory: Exercise Session 3 21 May 2012

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Problem 1: Lorentz covariant quantization of photon field in the light-cone basis and the physical space of states

We saw in lectures that after fixing to the Lorentz gauge the photon field has the form:

$$A_\mu(x) = \int \tilde{dk} \sum_\lambda \epsilon_\mu^{(\lambda)}(\vec{k}) \left[a_\lambda(\vec{k}) e^{-ik.x} + a_\lambda^\dagger(\vec{k}) e^{ik.x} \right] . \tag{1}$$

We chose a basis for the four polarization vectors $\epsilon_\mu^{(\lambda)}(\vec{k})$ corresponding to time-like, longitudinal and transverse polarizations. Another convenient basis is the light-cone basis with (λ takes values $k, \bar{k}, 1, 2$):

$$\epsilon_\mu^{(\lambda)}(\vec{k}) = \left\{ \frac{1}{\sqrt{2}} \frac{k_\mu}{|\vec{k}|}, \frac{1}{\sqrt{2}} \frac{\bar{k}_\mu}{|\vec{k}|}, \begin{pmatrix} 0 \\ \vec{e_1} \end{pmatrix}, \begin{pmatrix} 0 \\ \vec{e_2} \end{pmatrix} \right\} , \tag{2}$$

where

$$k^\mu \equiv (|\vec{k}|, \vec{k}), \quad \bar{k}^\mu \equiv (|\vec{k}|, -\vec{k}), \quad \text{and} \quad \vec{e_i}.\vec{e_j} = \delta_{ij}, \quad \vec{e_i}.\vec{k} = 0 \ \text{ for } \ i,j = 1,2 . \tag{3}$$

(a) Compute the commutators ($\lambda, \sigma = k, \bar{k}, 1, 2$)

$$\left[a_\lambda(\vec{k}), a_\sigma^\dagger(\vec{k'}) \right] \tag{4}$$

using

$$a_\mu(\vec{k}) = \sum_\lambda \epsilon_\mu^{(\lambda)}(\vec{k}) a_\lambda(\vec{k}) \tag{5}$$

and

$$\left[a_\mu(\vec{k}), a_\nu^\dagger(\vec{k'}) \right] = -\eta_{\mu\nu} \tilde{\delta}(\vec{k} - \vec{k'}) . \tag{6}$$

Hint: Use the scalar product

$$g^{\lambda\sigma} \equiv \epsilon_\mu^{(\lambda)}(\vec{k}) \eta^{\mu\nu} \epsilon_\nu^{(\sigma)}(\vec{k}) . \tag{7}$$

(b) The one-particle Fock space $H^{(1)}$ is the vector space spanned by the states $a_\lambda^\dagger(\vec{k})|0\rangle$. Determine explicitly the subspace $H_{inv}^{(1)} \subset H^{(1)}$ of states $|\psi\rangle$ which satisfy the Gupta-Bleuler condition:

$$\partial_\mu A^{\mu(+)} |\psi\rangle = 0, \quad \text{where} \ \ A_\mu^{(+)} \equiv \int \tilde{dk} \, a_\mu(\vec{k}) e^{-ik.x} . \tag{8}$$

(c) Compute the scalar product of the state $a^\dagger_{\bar{k}}|0\rangle \in H^{(1)}_{inv}$ with an arbitrary vector in $H^{(1)}_{inv}$. How would you define a physically reasonable state space $H^{(1)}_{phys}$?

Problem 2: Lorentz covariant quantization of the photon field with a general gauge-fixing term

An action which gives rise to Maxwell's equations in the Lorentz gauge is:

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \alpha \left(\partial . A \right)^2 \right] = \frac{1}{2} \int d^4x \, A_\mu \left[\Box \eta^{\mu\nu} + (\alpha - 1) \partial^\mu \partial^\nu \right] A_\nu \, . \tag{9}$$

In lectures and Problem 1 we used what is called the Feyman gauge, with $\alpha = 1$.

(a) Determine the propagator in momentum space. What happens when $\alpha = 0$?

(b) Write down the equations of motion for $A_\mu(x)$ and hence show that $A_\mu(x)$ satisfies the equations:

$$\Box \partial . A = 0 \quad \text{and} \quad \Box \Box A_\mu = 0 \tag{10}$$

(c) Solve the equations (10) subject to the constraint $A_\mu = A^\dagger_\mu$ and so that the equations of motion hold.

Hint: With the ansatz $A_\mu(x) = A_\mu(t) e^{i\vec{k}.\vec{x}}$, the general solution to

$$\left(\frac{d^2}{dt^2} + p^2 \right)^2 A_\mu(t) = 0 \tag{11}$$

is $A_\mu(t) = (a^\dagger_\mu + t\, b^\dagger_\mu) e^{ipt} + (a_\mu + t\, b_\mu) e^{-ipt}$, for some parameters $a^\dagger_\mu, a_\mu, b^\dagger_\mu, b_\mu$.

(d) Given that $[iP_\mu, A_\nu] = \partial_\mu A_\nu$, compute the commutators $\left[P_0, a^\dagger_\mu(\vec{k}) \right]$ and $\left[P_0, a^\dagger_\lambda(\vec{k}) \right]$ (where $\lambda = k, \bar{k}, 1, 2$). Can the Hermitian operator $P^0 = H$ be diagonalized on the one-particle Fock space?

(e) The commutation relations for creation and annihilation operators read $(i, j = 1, 2)$:

$$\left[a_i(\vec{k}), a^\dagger_j(\vec{k'}) \right] = \delta_{ij} \tilde{\delta}(\vec{k} - \vec{k'}) \tag{12}$$

$$\left[a_k(\vec{k}), a^\dagger_{\bar{k}}(\vec{k'}) \right] = -\frac{\alpha + 1}{2\alpha} \tilde{\delta}(\vec{k} - \vec{k'}) \, . \tag{13}$$

Determine the physical state space, in the same manner as Problems 1 (b) and (c).